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Contact Structure with Basic Potentials

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This paper defines basic potentials for contact structure. Contact manifolds that admit a basic potential are shown to have an additional foliated structure of co-dimension 1. The properties of this new foliation, and its relation to the characteristic vector field, are explored.

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1. INTRODUCTION

The terms *contact structure* and *contact manifolds* are used by authors to refer to a variety of different, but closely related, structures all of which, in one way or another, play a fundamental role in mathematical physics. In its widest sense, contact structure embraces a number of more specialised sub-geometries, and also some closely related structures that may be regarded as alternative generalisations of them. In many modelling problems, this broadest definition of contact structure is too general. Models can often be formulated more naturally and conveniently on contact manifolds of more specialised type. One such model is defined by a global 1-form α of maximal rank, called the *contact 1-form*. In fact, this more specialised structure turns up so frequently that some older authors seem unaware of the existence of any other. For them, this special type *is* contact structure.

It is true that contact structure no longer holds pride of place in analytical mechanics. It is now known that time-dependent Hamiltonian mechanics is more appropriately formulated in terms of *co-symplectic geometry* (Cantrijn *et al.*, 1992; de León and Rodrigues, 1989). This is a $(2n + 1)$-dimensional manifold M

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equipped with a closed 2-form ω of maximal rank, together with a closed 1-form α such that, at each point $x \in M$, $\alpha \wedge \omega^n \neq 0$. Comparing the definitions, it is seen that there is a sense in which co-symplectic manifolds are opposite to the contact manifolds (Albert, 1989). They also differ in another very important way: while contact manifolds are Jacobi in nature, co-symplectic ones are Poisson (Dazord *et al.*, 1991; de León *et al.*, 1997; Ibort *et al.*, 1997). Nevertheless, contact and contact-like manifolds play an important role in a variety of fields other than mechanics. And, despite of their overthrow from priviledged position in analytical mechanics by co-symplectic structure, they continue to be important in that field also (Abraham and Marsden, 1978; Arnold, 1978; Godbillon, 1969). Their study thus continues to be of significance.

In this paper, we investigate some properties of a particular type of contact 1-form that does not appear to have been considered before. This is one in which its exterior derivative, the two form $d\alpha$, admits a basic potential. "Basic" here refers to the fact that the contact form α endows the manifold with a foliated structure of dimension 1. The existence of a basic potential endows the manifold with a second foliated structure, this time of co-dimension 1, which we explore.

In Section 2, we state the definition of contact structure we use in this work. This is necessary because of the wide variety of definitions, not all equivalent, found in the literature. It is not our intention to champion this definition. We have simply found it a useful starting point for our investigations. We then briefly review some of its most important consequences, and also add some non-standard definitions of our own which we use in the proofs of our results. In Section 3, we introduce the concept of basic potentials for contact structures, and deduce some of their properties. A special case, which we call basic potentials of Liouville type, is also briefly considered. Finally, in Section 4, we deduce some properties of the second foliated structure on the contact manifold introduced by the basic potentials.

2. CONTACT STRUCTURE

Not all authors are agreed on the definition of contact structure, or on the terminology they use. Blair, for example, acknowledges two definitions, which he calls respectively *contact structure (in the wider sense)* and *contact structure (in the restricted sense)* (Blair, 1976). Abraham and Marsden also acknowledge two definitions, and call the manifolds they define respectively *contact manifolds* and *exact contact manifolds* (Abraham and Marsden, 1978). Their exact contact manifold has a structure that coincides precisely with what Blair calls a contact structure in the restricted sense, but their contact manifold is somewhat less general than what Blair calls a manifold with a contact structure in the wider sense. Arnold adopts yet another definition in terms of "a smooth field of tangent hyperplanes satifying a non-degeneracy condition" (Arnold, 1978), while Liberman and Marle offer a complex of contact structures of differing degrees of generality which overlap in good measure with some of those mentioned above, but using completely different terminology (Liberman and Marle, 1987). It is not our object in this paper to compare and contrast these different defintions, nor to champion any one of them. Rather, we have chosen to cut this intricate Gordian knot by simply adopting a definition which is convenient for our work, and which we will now state.

The definition of contact structure that we will use in this paper is the one which Blair calls *contact structure (in the restricted sense)*, and which Liberman and Marle call *Pfaffian contact structure*. Our reasons for this are twofold. First, this type of contact structure is the simplest to define, and with which to work. Second, Gray has shown that, for orientable manifolds, Blair's contact structure in the wider sense is equivalent to his contact structure in the restricted sense (Gray, 1959). In this sense therefore, it almost (but not quite) captures the essence of contact structure in its most general definition. Besides, for a large number of authors, "contact structure" means precisely contact structure in Blair's restricted sense. Popular vote thus appears to be on its side.

This type of contact structure is defined as follows. Let M be a smooth differentiable manifold of odd dimension $2n+1$, with $n > 0$. A *contact structure* on M is then a global differential 1-form α with the property that

$$\Omega = \alpha \wedge d\alpha^n \neq 0 \tag{2.1}$$

everywhere on M. The exponent denotes the nth exterior power. The 1-form α is called the *contact form*. Note that Ω is a non-zero form of degree $(2n+1)$ on M, and so is a volume form. The manifold M is thus necessarily orientable. The classical theorem of Darboux asserts that, given any point $x \in M$, there exists a chart $\{U, \varphi\}$ on M with coordinate functions $\{z, x^i, p_i\}$, $i = 1, \ldots, n$, such that α takes the canonical form

$$\alpha = dz + p_i dx^i \tag{2.2}$$

The 1-form α defines at each point of M a $2n$-dimensional hyperplane D_x in T_xM. The set of all these hyperplanes is a subbundle D of TM and is called the *contact distribution*. Since, according to (2.1), $d\alpha \neq 0$, this distribution is not integrable. Blair comments that condition (2.1) means that "loosely speaking, D is as far from being integrable as possible." The 2-form $\omega = d\alpha$ is of rank $2n$, and is also everywhere non-zero on M. It is thus a kind of "vortex field" and defines at each point x in M a 1-dimensional subspace C_x of T_xM. Trivially, this 1-dimensional distribution on M is integrable, because it is 1-dimensional and/or because $d\omega = 0$. M is thus a foliation with leaves of dimension 1. It is easy to show that the distribution C, called the *characteristic distribution*, is transverse to D, and so equips T_xM at each point $x \in M$ with a natural direct sum structure. The transversality of C and D means that we can use the contact form α to "measure" vectors in C. Choosing at each point a vector of unit "length" defines an unique global vector field ξ on M, called the *characteristic vector field* of M. This vector field is thus defined by

$$i_\xi \alpha = 1, \qquad i_\xi \omega = 0 \tag{2.3}$$

The contact structure is said to be *regular* if ξ is a regular vector field on M. Note furthermore that ω, restricted to D, is symplectic. So, D is a symplectic vector bundle with base M. As a foliation, therefore, M is transversally symplectic.

Besides these well known objects and structures, we introduce also a bilinear form B defined by

$$B = \alpha \otimes \alpha + \omega \tag{2.4}$$

B is non-degenerate, but has no definite symmetry. It is a metric on C, and symplectic on D. Because it is non-degenerate, we can use it to define raising and lowering operators $\sharp$ and $\flat$. Its lack of symmetry mean that, strictly, we should distinguish left and right versions of these operators, but its simple properties on the subbundles C and D mean that these operators are easily related and we can dispose of one of the two sets. We therefore define $\flat$ as follows. For arbitrary vectors $X \in TM$ and given A, put

$$\flat A(X) = B(A, X) \tag{2.5}$$

Put differently, B is nothing but the vector bundle isomorphism

$$X \mapsto i_X \omega + (i_X \alpha)\, \alpha$$

Also, define $\sharp = \flat^{-1}$. It is easy to show that $\xi = \sharp\alpha$, and we will use these two symbols interchangeably, depending on which is more transparent in the context.

3. BASIC POTENTIALS

We now investigate some special properties of contact manifolds in which the 2-form ω admits a potential which is basic. Basic and semi-basic forms are discussed in detail in Liberman and Marle in the context of foliations defined by surjective submersions (Liberman and Marle, 1987), but the concepts are more general and can be transferred unchanged into the theory of general foliations. Of course, ω by definition has a potential, namely α. However α is not a basic form. Because basic forms are simpler than non-basic ones, it is desirable to investigate the implications of the existence of a basic replacement for α as a potential for ω.

Suppose ω has a basic potential. That is, there is on M a 1-form β such that $\omega = d\beta$, with properties $i_\xi \beta = 0$ and $\mathcal{L}_\xi \beta = 0$. Here, ξ is the characteristic vector field of the Pfaffian structure. In a Darboux chart for α, with coordinate functions

$\{z, x^i, p_i\}$, $i = 1, \ldots, n$, we have

$$\alpha = dz + p_i dx^i, \quad \omega = dp_i \wedge dx^i, \quad \xi = \frac{\partial}{\partial z}$$

and

$$\beta = \beta_i(x, p)dx^i + \tilde{\beta}^i(x, p)dp_i$$

where the coefficients β_i and $\tilde{\beta}^i$ are functions of the variables $\{x^1, \ldots, x^n, p_1, \ldots, p_n\}$ alone, and do not involve the variable z. By definition, α is everywhere non-zero, and it is obvious from these coordinate characterisations that

$$\theta = \alpha - \beta \tag{3.1}$$

is also everywhere non-zero. It is also closed, since $d\theta = d\alpha - d\beta = \omega - \omega = 0$. The 1-form θ thus defines a foliation $\{E_x\}$ of co-dimension 1 on M.

The following identities follow easily from the above definitions:

$$\sharp\beta \rfloor \beta = 0, \quad \sharp\beta \rfloor \omega = \beta, \quad \sharp\beta \rfloor \alpha = 0, \quad \sharp\beta \rfloor \theta = 0 \tag{3.2}$$

$$\mathcal{L}_{\sharp\beta}\alpha = \beta, \quad \mathcal{L}_{\sharp\beta}\beta = \beta, \quad \mathcal{L}_{\sharp\beta}\omega = \omega, \quad \mathcal{L}_{\sharp\beta}\theta = 0 \tag{3.3}$$

$$\sharp\theta \rfloor \beta = 0, \quad \sharp\theta \rfloor \omega = -\beta, \quad \sharp\theta \rfloor \alpha = 1 \tag{3.4}$$

$$\mathcal{L}_{\sharp\theta}\alpha = -\beta, \quad \mathcal{L}_{\sharp\theta}\beta = -\beta, \quad \mathcal{L}_{\sharp\theta}\omega = -\omega \tag{3.5}$$

$$[\sharp\alpha, \sharp\beta] = 0, \quad [\sharp\alpha, \sharp\theta] = 0, \quad [\sharp\beta, \sharp\theta] = 0 \tag{3.6}$$

Relations (3.2) to (3.5) show that the vector field $\sharp\beta$ is tangent to the foliation $\{E_x\}$, and that $\sharp\alpha$ is transverse to it.

Denote the flows of the vector fields $\sharp\alpha$, $\sharp\beta$ and $\sharp\theta$ respectively by $\{\phi_t\}$, $\{\psi_t\}$ and $\{\chi_t\}$. Relations (3.6) then show that

$$\phi_t = \psi_t \circ \chi_t = \chi_t \circ \psi_t \tag{3.7}$$

Further, $\sharp\theta$ is B-perpendicular to the leaves of $\{E_x\}$ while $\sharp\beta$ is tangent to them. The flow of the vector field $\sharp\beta$ thus preserves the leaves of $\{E_x\}$.

Now, let E be a leaf of $\{E_x\}$. Denote by β_E and ω_E the forms induced on E by β and ω respectively. Then, from (3.3)

$$\mathcal{L}_{\sharp\beta}\beta_E = \beta_E$$

$$\mathcal{L}_{\sharp\beta}\omega_E = \omega_E$$

The 2-form ω_E on E is symplectic and so can be used to define a raising operator $\sharp_S$ on E. This operator has the property

$$\sharp\beta = \sharp_S\beta_E$$

LIOUVILLE-TYPE BASIC POTENTIALS:

We say that β is a basic potential of Liouville type if the zero set of β,

$$S_\beta = \{u \in M : \beta_u = 0\}$$

is an $(n+1)$-dimensional submanifold, and if at each point $u \in S_\beta$ there is a Darboux coordinate chart for β with coordinates (z, x^i, p_i) in which

$$\beta = p_i \, dx^i$$

that is, β has rank $2n$. In this chart, S_β is the manifold $p_i = 0$, and $\sharp\beta = p_i \, \partial/\partial p_i$. Thus S_β is a maximal isotropic submanifold for ω. The field $\sharp\alpha$ is tangent to S_β and so induces a flow on S_β. This flow preserves the leaves of the foliation of the closed non-zero form θ restricted to S_β.

We shall discuss the dynamics of $\sharp\alpha$ on the submanifold S_β in the next section. But first we establish the following result. Since $\sharp\alpha$ is the characteristic field, we have, in these coordinates,

$$\sharp\alpha = c \frac{\partial}{\partial z}$$

where c is a function of (z, x^i, p_i). But by (3.6)

$$[\sharp\alpha, \sharp\beta] = 0$$

so that

$$\mathcal{L}_{\sharp\beta} \, c = 0$$

This means that c is homogeneous of degree zero in the p_i, and since it is smooth, must be independent of p_i. The vector field $\sharp\alpha$ is therefore uniquely determined near the submanifold S_β by its values on S_β. If the manifold M and the fundamental form α are (real) analytic, then $\sharp\alpha$ is completely determined in all M by its values on S_β.

4. STRUCTURE OF THE FOLIATION $\{E_X\}$

Consider now the structure of the foliation $\{E_x\}$. Our discussion is modelled on that in Hector and Hirsch (1983), pp. 152–157. In this section, we assume that $\sharp\alpha$, $\sharp\beta$ and $\sharp\theta$ are complete vector fields on M.

There are then only three possibilities for $\{E_x\}$:

(a) $\theta = df$, in which case $\{E_x\}$ is a (locally trivial) fibration over $\mathbb{R}$,
(b) The group Π_θ of periods of θ is cyclic and $\{E_x\}$ is a locally trivial fibration over S^1,
(c) Π_θ is dense in $\mathbb{R}$ and $\{E_x\}$ is minimal, that is, all its leaves are dense in M.

We are thus led to the following proposition.

Proposition 1. *If M is compact then $\{E_x\}$ is minimal.*

Proof: The restriction ω_F of ω to any leaf F of $\{E_x\}$ defines a symplectic form on F which is exact, since $d\alpha_F = \omega_F$. Hence F is not compact. This precludes options (a) and (b) listed above, so $\{E_x\}$ must be minimal. □

Consider now the flow $\{\phi_t\}$ of the vector field $\sharp\alpha$ for the cases (a) and (b) listed above.

Proposition 2. *Case (a): $\{\phi_t\}$ has no periodic orbits and the action of $\{\phi_t\}$ on the leaves of $\{E_x\}$ is simply transitive.*
Case (b): The flow induced on the basis S^1 is transitive.

Proof: Case (a): Suppose $\phi_t(E_x) = E_x$ for some t. Then the flow induced on the base space $\mathbb{R}$ has a periodic orbit and must thus have a stationary point, say s in $\mathbb{R}$. Thus the leaf F which projects onto s is stationary, that is, for all $t \in \mathbb{R}$

$$\phi_t(F) = F$$

But this this is impossible since $\sharp\alpha$ is transversal to F, ($\sharp\alpha \rfloor \theta \neq 0$). The flow induced on the basis $\mathbb{R}$ is therefore transitive and so the flow $\{\phi_t\}$ acts simply transitively on the leaves $\{E_x\}$.

Case (b): Proved similarly. □

Finally we note that if we take a regular covering

$$\pi : \tilde{M} \to M$$

such that $\tilde{\theta} = \pi * \theta$ is exact, then

$$\tilde{\theta} = df$$

Further, $\tilde{M}$ is naturally equipped with the pullbacks $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\omega}$ which have the properties $\tilde{\theta} = \tilde{\alpha} - \tilde{\beta}$ and $\tilde{B} = \tilde{\alpha} \otimes \tilde{\alpha} + \tilde{\omega}$. On $\tilde{M}$ we therefore have case (a) listed above. The orbits of $\sharp\tilde{\alpha}$ are thus regular coverings of the orbits of $\sharp\alpha$.

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